J. Fluid Mech. (1965), *vol.* 21, *part* 2, *pp.* 345–358
Printed in Great Britain

The structure of non-linear cellular solutions to the Boussinesq equations

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(Received 13 May 1964)

A model equation is constructed whose analysis reveals the same features, including stable hexagonal cells, as analysis of genuine approximate equations for the thermal convection problem. Taking advantage of the relative simplicity of the calculations an attempt is made to clarify certain procedures customarily used in non-linear stability theory; in particular, the basis for the usual formal expansions and the appropriateness of neglecting terms of fourth and higher order are discussed. It is demonstrated that a whole class of equations leads to hexagonal cells, thereby giving confidence that results on convection cells found elsewhere in idealized situations will remain valid when more realistic situations are studied.

1. Introduction

Papers by Palm (1960), Segel & Stuart (1962), and Segel (1965) have made progress in explaining the hexagonal cell shape frequently observed when a horizontal layer of fluid becomes unstable as a result of being heated from below. (We refer to these papers as I, II, and IV. The present paper is referred to as III in IV.) The key to this progress is the possibility of reducing the set of non-linear partial differential equations governing the physical problem to a set of non-linear ordinary differential equations called the *amplitude equations.* Unfortunately, considerable manipulation is necessary to do this [the original governing equations take nine lines to write down in I] so that the reader is apt to become discouraged before he reaches the principal arguments. It is not possible to circumvent this difficulty by taking the amplitude equations on faith because a general appreciation of how these equations are derived is required for an understanding of various arguments in IV.

We have therefore thought it advisable to analyse a simple 'model equation' that retains to a high degree the characteristics of the physical problem under investigation. The analysis is carried out by a successive-approximation method which, while in essence only a reformulation of the formal series-expansion method customarily used, has the advantage of leading more naturally to the amplitude equations. We discuss in detail why it is appropriate to neglect fourth- and higher-order terms in these equations, a point which frequently seems not fully appreciated.

In order that they should have equilibrium solutions corresponding to hexagonal cells, the amplitude equations must have coefficients which satisfy certain relations. It was shown in I and II, for amplitude equations corresponding to the Boussinesq equations with a certain law of viscosity variation and 'free-free' boundary conditions, that these relations are satisfied. The question arises as to whether this is an accidental consequence of the various particular assumptions made or whether certain general features of the full governing equations are responsible for the behaviour of their solutions. That the latter is true is indicated by the success of the model equation in reproducing the qualitative features of the full problem. The investigation of § 4 confirms that certain overall properties of the governing partial differential equations are sufficient to insure the presence of hexagonal equilibrium points. (A by-product of this investigation is the fairly easy extension of the two-disturbance analysis of I and II to a six-disturbance analysis in IV. Otherwise, such an extension would be very tedious.) We therefore have reason to believe that the main conclusions of I, II, and IV will be unaltered when more accurate equations and boundary conditions are studied. This has been confirmed by Davis (1964) who used the results of § 4.

2. Analysis of the model equation

Burgers (much of whose work is summarized in Burgers 1948) was probably the first to exploit the use of a model equation in fluid mechanics. His study of various simplified versions of the Navier–Stokes equations gives a clear qualitative picture, relatively free of computational obscurities, of many phenomena which must appear in the turbulent motion of actual fluids. Hopf (1950) has been able to characterize rigorously the behaviour of simple model 'flows' as the 'viscosity' tends to zero. Several authors have investigated, largely numerically, sets of non-linear partial differential equations representing very highly simplified models of meteorological flows. One of the most interesting of these studies is that of Lorenz (1963). Finally, just as the present work was completed the author learned that Eckhaus (1962) has illustrated his approach to non-linear stability analyses on a model equation. One of our goals is to illustrate a non-linear stability analysis on a model equation, but we also seek fuller physical understanding of thermal convection so, unlike Eckhaus, we have incorporated into our model features analogous to the variation of viscosity with temperature and the ambiguity of horizontal structure which are essential to a discussion of the preferred pattern for cellular thermal convection. J. T. Stuart has independently made various unpublished calculations using a similar model.

We consider the following model equation for $W(x,y,z,t)$ when $0 \leqslant z \leqslant 1$:

$$L(W)-N(W) = 0, \tag{1}$$

$$L(W) \equiv -W_t+\Delta^3 W+2b\cos\pi z\, W - S\Delta_1 W, \tag{2}$$

$$N(W) \equiv (WW_z)_{zz}. \tag{3}$$

[S is a positive constant, b is a constant whose absolute value is small compared to unity, $\Delta W = \Delta_1 W + W_{zz}$ and $\Delta_1 W \equiv W_{xx}+W_{yy}$]. We impose the boundary conditions

$$W = W_{zz} = W_{zzzz} = 0 \quad \text{at} \quad z = 0, 1; \quad W \text{ bounded as } x^2+y^2 \to \infty. \tag{4}$$

We do not impose initial conditions because our interest will be concentrated on steady solutions to (4) and certain time-dependent transitions from one such solution to another.

We are thinking of the following analogies with the thermal convection problem: x and y are the 'horizontal co-ordinates' and z is the 'vertical co-ordinate'. W is the 'vertical velocity', S is the 'Rayleigh number' which is slowly increased until instability sets in, and b is the dimensionless measure of 'viscosity variation with temperature' whose presence gives rise to terms which play a central role in I, II, and IV.

The model equation and boundary conditions have a solution $W \equiv 0$. As a first step in a stability analysis of this solution we neglect non-linear terms and look for a small perturbation W_1 which satisfies $L(W_1) = 0$. Seeking a 'normal mode' solution to this equation of linear stability theory we are led to

$$\left.\begin{aligned} W_1(x,y,z,t) &= \exp(\epsilon t)\cos mx \cos ny\, g(z), \\ g(z) &= A_1 \sin \pi z + bA_2 \sin 2\pi z + O(b^2), \end{aligned}\right\} \tag{5}$$

where ϵ, m, n, A_1, and A_2 are constants. W_1 already satisfies the homogeneous boundary conditions; an eigenvalue equation comes from satisfying $L(W_1) = 0$ by setting the respective coefficients of $\sin \pi z$ and $\sin 2\pi z$ equal to zero. In the resulting pair of homogeneous equations for A_1 and A_2 we require that the determinant of the coefficients be zero, to avoid the trivial solution $A_1 = A_2 = 0$, and find

$$\left.\begin{aligned} \epsilon &= S\pi^2\alpha^2 - \pi^6(\alpha^2+1)^3 + b^2K(\alpha) + O(b^4), \\ K^{-1} &\equiv \pi^6[(\alpha^2+4)^3 - (\alpha^2+1)^3]. \end{aligned}\right\} \tag{6}$$

The x and y wave-numbers m and n occur here only in the overall wave-number α, defined by $\pi^2\alpha^2 \equiv m^2+n^2$, since x and y appear only in the combination $\partial^2/\partial x^2 + \partial^2/\partial y^2$. When $\epsilon = 0$ (the marginal case dividing stability from instability) we obtain

$$S = S_c(\alpha) = \pi^4(\alpha^2+1)^3\alpha^{-2} - Kb^2\pi^{-2}\alpha^{-2} + O(b^4). \tag{7}$$

The minimum value of $S_c(\alpha)$ occurs at the critical overall wave-number α_c given by

$$\alpha_c^2 = 0{\cdot}5 - 0{\cdot}0033\pi^{-12}b^2 + O(b^4), \tag{8}$$

giving for S_c, the minimum critical value of $S_c(\alpha)$,

$$S_c \equiv S_c(\alpha_c) = \tfrac{27}{4}\pi^4 - 0{\cdot}023\pi^{-8}b^2 + O(b^4). \tag{9}$$

The curve of marginal stability is given in figure 1; this curve is virtually the same as the corresponding curve found in the linearized theory of cellular convection. The $O(b^2)$ decrease of α_c and S_c found here is also found for the corresponding quantities calculated from the full equations. Finally, the two useful relationships

$$\epsilon = \pi^2\alpha^2(S - S_c), \quad A_2 = KA_1, \tag{10}$$

are easily derived.

We turn now to the complete non-linear equation and attempt to find solutions which are small for all time. The most natural approach is to take a linear-theory solution as a first approximation W_1 and to find further approximations using $L(W_n) = N(W_{n-1})$. This will not lead to solutions of the desired type, however, for if $W_1 \sim \exp(\epsilon t)$, then $W_2 \sim \exp 2\epsilon t$, etc., and the ratio of the $(n+1)$st approximation to the nth has the undesirable property of becoming indefinitely large as time increases, so at no stage do we obtain an approximation uniformly valid for $-\infty < t < \infty$. The 'natural' approach can be made satisfactory by means of certain special devices, but a referee has pointed out that a straightforward procedure results from rewriting the basic equation as

$$\mathscr{D}(W) = \mathscr{M}(W),$$

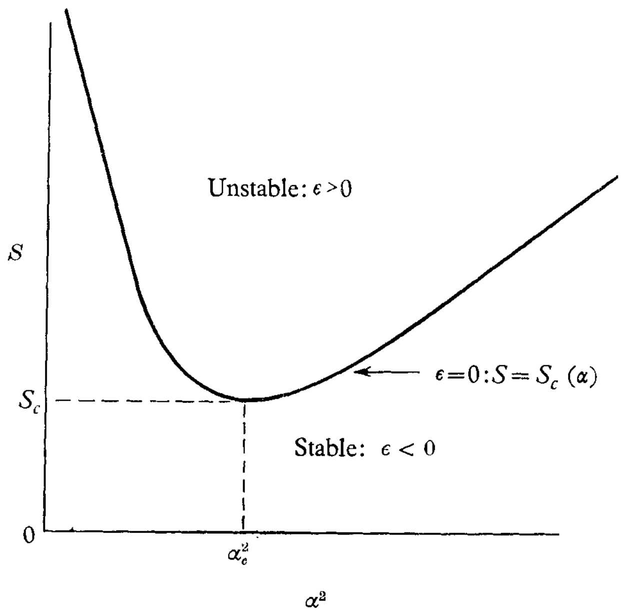

FIGURE 1. Model equation: Critical S ('Rayleigh number') *vs* square of overall wave-number according to linear stability theory. Disturbance amplitude at time t is proportional to $\exp(\epsilon t)$.

where

$$\mathscr{D}(W) \equiv \Delta^3 W - S_c \Delta_1 W + 2b \cos \pi z W, \quad \mathscr{M}(W) \equiv W_t + (S - S_c)\Delta_1 W + (W W_z)_{zz},$$

and solving successively

$$\mathscr{D}(W_1) = 0; \quad \mathscr{D}(W_n) = \mathscr{M}(W_{n-1}) \quad (n = 2, 3, \ldots). \tag{11}$$

This procedure explicitly requires not only small non-linear terms but small $S - S_c$ (a natural requirement since amplitudes approach zero as $S \to S_c$, so small $S - S_c$ will imply small amplitudes) and small W_t (a natural requirement since $W_t \sim (S - S_c) W$ by linear theory).

For simplicity we first discuss the iteration for the special case $b = 0$. In W_1, to parallel the analysis of I and II, we consider the (x, y)-dependence to be composed of two particular terms of the same over-all wave-number:

$$W_1 = \phi(x, y, t)\, g(z), \quad \phi = Y(t) \cos \tfrac{\sqrt{3}}{2}\pi\alpha x \cos \tfrac{1}{2}\pi\alpha y + Z(t) \cos \pi\alpha y. \tag{12}$$

$\mathscr{D}(W_1) = 0$ is satisfied for arbitrary $Y(t)$ and $Z(t)$ if

$$g(z) = \sin \pi z$$

and S_c is the same as the linear theory result (9), with $b = 0$. W_2 then satisfies

$$\mathscr{D}(W_2) = \mathscr{M}(W_1) \equiv [Y' - \pi^2\alpha^2(S-S_c)\,Y]\cos\tfrac{\sqrt{3}}{2}\pi\alpha x\cos\tfrac{1}{2}\pi\alpha y\, g(z) + [Z' - \pi^2\alpha^2(S-S_c)\,Z]\cos\pi\alpha y\, g(z) + \Phi(x,y,t)\,G(z),$$

where
$$G(z) = -2\pi^3\sin 2\pi z, \quad \Phi = \phi^2 = \sum_{m=0}^{4}\Phi_m; \tag{13}$$

$$\left.\begin{aligned} \Phi_0 &= \tfrac{1}{4}Y^2 + \tfrac{1}{2}Z^2, \quad \Phi_2 \equiv 0,\\ \Phi_1 &= \tfrac{1}{4}Y^2\cos\pi\alpha y + YZ\cos\tfrac{\sqrt{3}}{2}\pi\alpha x\cos\tfrac{1}{2}\pi\alpha y,\\ \Phi_3 &= \tfrac{1}{4}Y^2\cos\sqrt{3}\,\pi\alpha x + YZ\cos\tfrac{\sqrt{3}}{2}\pi\alpha x\cos\tfrac{3}{2}\pi\alpha y,\\ \Phi_4 &= \tfrac{1}{4}Y^2\cos\sqrt{3}\,\pi\alpha x\cos\pi\alpha y + \tfrac{1}{2}Z^2\cos 2\pi\alpha y. \end{aligned}\right\} \tag{14}$$

[Note that all terms in Φ_m have over-all wave-number $\sqrt{m}\,\alpha$.]

As $\mathscr{D}$ is time-independent and has constant coefficients, W_2 is easily seen to be given by

$$\left.\begin{aligned} W_2 &= G\sum_{m=0}^{4} D_m\Phi_m + W_1,\\ D_m^{-1} &\equiv -(m\pi^2\alpha^2 + 4\pi^2)^3 + m\pi^2\alpha^2 S_c, \end{aligned}\right\} \tag{15}$$

if terms proportional to solutions of the homogeneous problem are removed by taking

$$Y' = \epsilon Y, \quad Z' = \epsilon Z, \quad \epsilon = \pi^2\alpha^2(S-S_c). \tag{16}$$

Note that in (15) we have added W_1, the appropriate solution of the homogeneous equation, to a particular solution of $\mathscr{D}(W_2) = \mathscr{M}(W_1)$. Note also that, from (16), if we stop at this stage then $Y = Z = \exp(\epsilon t)$ where the value of ϵ is in agreement with our earlier linear theory result (10). To find W_3 we compute $\mathscr{M}(W_2)$ which has the same linear and quadratic terms in Y and Z as $\mathscr{M}(W_1)$, fourth-order terms which will be seen to be negligible, and third-order terms among which are

$$(RY^3 + PYZ^2)\cos\tfrac{\sqrt{3}}{2}\pi\alpha x\cos\tfrac{1}{2}\pi\alpha y\sin\pi z \tag{17a}$$

and
$$(\tfrac{1}{2}PY^2Z + R_1Z^3)\cos\pi\alpha y\sin\pi z, \tag{17b}$$

where the three constants

$$\left.\begin{aligned} R &\equiv -\tfrac{1}{16}\pi^6(4D_0 + 2D_1 + 2D_3 + D_4),\\ P &= -\tfrac{1}{2}\pi^6(D_0 + D_1 + D_3), \quad R_1 = -\tfrac{1}{4}\pi^6(2D_0 + D_4), \end{aligned}\right\} \tag{18}$$

are positive at least for $|S - S_c(\alpha)|$ small, i.e. for ϵ small. Expressions (17a, b) are further terms proportional to eigenfunctions of $\mathscr{D}$. We will call such terms *replicating terms*. At any stage in the iteration, the net coefficient of the replicating terms must be zero. This is because of the well known fact that if A is a self-adjoint linear operator [differential operator plus boundary conditions] the inhomogeneous equation $A(\psi) = F$ will have a solution only if F is orthogonal to the solution Ψ of $A(\Psi) = 0$. In computing W_3, the requirement that the coefficients of the replicating terms vanish yields

$$Y' = \epsilon Y - RY^3 - PYZ^2, \quad Z' = \epsilon Z - \tfrac{1}{2}PY^2Z - R_1Z^3. \tag{19}$$

The main ideas needed in the successive approximations method having been presented, we will summarize the procedure.

(i) A given approximation W_n is found from the previous approximation W_{n-1} by solving the linear partial differential equation

$$\mathscr{D}(W_n) = \mathscr{M}(W_{n-1}). \tag{20}$$

(ii) The method of undetermined coefficients gives particular solutions of (20) corresponding to terms in $\mathscr{M}$ whose (x, y, z) dependence is *different* from the (x, y, z) dependence of the terms in the first (linearized) approximation W_1.

(iii) A term in $\mathscr{M}$, say

$$C[A_j(t)]^p\,[A_k(t)]^q \cos mx \cos ny\, g(z),$$

whose (x, y, z) dependence as the *same* as a term

$$A_i(t) \cos mx \cos ny\, g(z)$$

in the first approximation W_1 requires the presence of a term $-CA_j^p A_k^q$ in the ordinary differential equation governing $A_i(t)$:

$$A_i' = \epsilon A_i + \ldots - CA_j^p A_k^q \ldots.$$

In this manner the original non-linear partial differential equation is reduced to a set of non-linear ordinary differential equations for the amplitude functions $A_i(t)$. Let us call these the *amplitude equations*. Much is now known about sets of non-linear ordinary differential equations so that this approach reduces the original problem to a tractable state while retaining its non-linear character. Since the qualitative behaviour of the original equation is governed by the amplitude equations, the rule by which these equations are formulated is very important. This is given in (iii) above and will be termed *amplitude modification through replication* since the linearized amplitude equation $A' = \epsilon A$ is modified through the presence of replicating higher-order terms. (It was thought advisable in this special context to use the word 'replication', rather than more common synonyms like 'repetition' or 'reproduction'. A term making use of the word 'resonance' was not used since resonance is usually associated with large amplitudes while in the case considered here, for example, third-order replication will be seen to cause the equilibration of solutions which grow without bound according to linear theory.)

Non-linear analyses in previous papers have not used the successive-approximation method presented here, but have formally expanded the solution in a Fourier series with time-varying coefficients, which in turn are expressed as series in powers of the several time-varying amplitudes of the linear disturbance modes (Stuart 1961). A successive-approximation procedure has always been the implicit basis for choosing the correct power-series expansions of the Fourier coefficients so it has been thought worthwhile to present this procedure explicitly. Furthermore, the author has found a successive-approximation analysis of a model equation the swiftest way to give to those unfamiliar with non-linear stability theory a reasonably secure understanding of its pivotal feature, amplitude modification through replication.

The formal expansion and successive-approximation procedures are alternative approaches to similar, but not identical, expansions. For example, the successive-approximation procedure yields a term

$$-[-64\pi^6(a^2+1)^3+4\pi^2\alpha^2S_c]^{-1}\,\pi^3Z^2\cos 2\pi\alpha y\sin 2\pi z$$

in W_2 [see (15)] but the corresponding term according to the formal expansion method turns out to be

$$-[-2\epsilon-64\pi^6(\alpha^2+1)^3+4\pi^2\alpha^2S]^{-1}\pi^3Z^2\cos 2\pi\alpha y\sin 2\pi z$$

or, using (10),

$$-[-64\pi^6(\alpha^2+1)^3+2\pi^2\alpha^2(S+S_c)]\,\pi^3Z^2\cos 2\pi\alpha y\sin 2\pi z.$$

The corresponding terms differ by an $O(S-S_c)$ or 'higher-order' amount, which is not alarming since although there is presumably a unique solution of the type sought, there is not a unique nth approximation to this solution. There is no unique way to set up the successive-approximation procedure. The iteration

$$(-\epsilon+\Delta^3-S\Delta_1)\,W_n = (\partial/\partial t-\epsilon)\,W_{n-1}+\tfrac{1}{2}(W_{n-1}^2)_{zzz},$$

for example, gives uniformly valid approximations different from those already mentioned. Indeed the original iteration procedure $L(W_n) = M(W_{n-1})$ is a possible one but was discarded because it led to non-uniformly-valid approximations containing powers of $\exp(\epsilon t)$ rather than powers of quantities like the solution of (19) with $Y \equiv 0$, i.e.

$$[\epsilon C\,e^{2\epsilon t}/(1+R_1C\,e^{2\epsilon t})]^{\frac{1}{2}},\quad C \text{ an arbitrary constant,}$$

generated by $\mathscr{D}(W_n) = \mathscr{M}(W_{n-1})$. The different iteration procedures correspond to different arrangements of an infinite-series solution. The formal expansion method and our iteration procedure (11) both lead to approximations uniformly valid in time, but these different approximations may be of varying effectiveness as functions of S. The results of (11) are closely related to expansions in powers of $S-S_c$ like the steady-state solutions of the Malkus–Veronis type discussed in IV, but there is evidence that the formal expansion method produces expansions more nearly like those in powers of $(S-S_c)/S$ shown superior by Kuo (1961).

For simplicity in exposition we set $b = 0$ in deriving the amplitude equations (19). The effect of a non-zero b (analogous to a non-zero variation of viscosity with temperature) is to introduce second-order terms into (19). With $b \neq 0$, $g(z)$ in (12) becomes

$$\sin\pi z+bK\sin 2\pi z+O(b^2)$$

so $G(z)$ of (14) now is

$$G(z) = -2\pi^3\sin 2\pi z+\tfrac{1}{2}\pi^3bK(\sin\pi z-27\sin 3\pi z)+O(b^2),$$

which can be written $\quad G(z) \equiv -\tfrac{3}{2}\pi^3bKg(z)+g_1(z),$

where $g_1(z)$ is orthogonal to $g(z)$, the function giving the linear-theory z-dependence:

$$\int_0^1 g(z)\,g_1(z)\,dz = 0+O(b^2).$$

With $b = 0$, the (x, y) dependence of the linear term is replicated by the Φ_1 terms in (13) but there is no z-replication since $\sin 2\pi z$ is orthogonal to $\sin \pi z$. With $b \neq 0$, as we have just shown, there is z-replication due to the presence of the $g(z)$ term in the expression for $G(z)$. Because of this second-order replication, if we write

$$a \equiv -\tfrac{3}{2}\pi^3 bK,$$

the amplitude equations now read

$$Y' = \epsilon Y - aYZ - RY^3 - PYZ^2, \tag{21}$$

$$Z' = \epsilon Z - \tfrac{1}{4}aY^2 - R_1 Z^3 - \tfrac{1}{2}PY^2 Z. \tag{22}$$

There are also $O(b)$ corrections to R, R_1 and P, but these can be neglected.

Equations (21) and (22) are *identical* with those discussed completely in II so all the results derived there are true here. In particular there exist hexagonal equilibrium points (where $Y = \pm 2Z$) which are stable when ϵ is sufficiently small. [Note that although the expressions, in terms of the basic parameters of the problem, of the coefficients ϵ, a, R, R_1 and P are different in the two cases, (18) shows that the vital relation $P = 4R - R_1$ holds here as well as in II.]

A word might now be said about the choice of the model equation. Aside from the quantity proportional to b, the linear terms were chosen so that the neutral stability curve for Bénard cells would be reproduced. The term $2b \cos \pi z W$ was added to model the effect of viscosity variation with temperature. Examination of the actual non-linear terms [cf. (30)] indicated that their effect would be mirrored by $(W^2)z$ but calculations showed that the additional two z-differentiations were necessary. Otherwise second-order terms in the amplitude equations would not appear, even for non-zero b.

Finally, we note that the linearized model equation is self-adjoint but the original linearized equation [I, equation (5.1)] is not. (The non-self-adjointness occurs because of the variation of viscosity with temperature and is connected only with z-differentiations.) When the linearized equation is non-self-adjoint the successive approximations procedure goes through just as before except that replicating terms are those whose (x, y, z) dependence is the same as that of the solution to the *adjoint* of the linearized problem.

3. Comments on the validity of truncating the amplitude equations

We have implicitly assumed that correct qualitative behaviour is obtained if terms through those of third order are kept in the amplitude equations. To see that this is so, we multiply equation (21) by Y and equation (22) by Z, obtaining the energy-type equations

$$\left.\begin{aligned}(\tfrac{1}{2}Y^2)' &= \epsilon Y^2 - aY^2 Z - Y^2(RY^2 + PZ^2),\\ (\tfrac{1}{2}Z^2)' &= \epsilon Z^2 - \tfrac{1}{4}aY^2 Z - Z^2(\tfrac{1}{2}PY^2 + R_1 Z^2).\end{aligned}\right\} \tag{23}$$

The highest-order terms on the right-hand side are clearly stabilizing as they cause a decrease in the magnitudes of Y and Z. (These highest-order terms in (23) are fourth order, but they arise from third-order terms in the amplitude equations.) The third-order terms in (23) are stabilizing if $aZ > 0$ but destabilizing if $aZ < 0$ so that regardless of the sign of a these terms will exert a destabilizing influence in half of the (Y, Z)-(phase) plane. The lowest-order terms are stabilizing if $\epsilon < 0$ and destabilizing if $\epsilon > 0$, as predicted by linear stability theory. Consequently, if linear theory predicts instability ($\epsilon > 0$) we must take into consideration third-order terms in the amplitude equations (fourth-order terms in (23)) or else some solutions will grow without bound, thereby violating the assumptions of our theory that Y and Z are small for all time. If linear theory predicts stability ($\epsilon < 0$) it is consistent to neglect all non-linear terms if the initial values of Y and Z are sufficiently small: the correct qualitative prediction of decay with time results. On the other hand, if Y^2Z is initially sufficiently large and Z has the appropriate sign then second-order theory predicts (inconsistently with its perturbation nature) that these disturbances will grow without bound. Once again, if third-order terms are considered we can obtain a consistent picture of instabilities due to finite-amplitude effects growing to an equilibrium amplitude but still remaining uniformly small for all time. Computation of the equilibrium values of Y and Z (from setting $Y' = Z' = 0$) shows that $Y(t)$ and $Z(t)$ will in fact be small for all time if $\epsilon^{\frac{1}{2}}$ and a are small. We must therefore make these assumptions if the perturbation method is to make sense. All our remarks apply to the Boussinesq amplitude equations and so provide the reasons for the physical assumptions in I, II and IV that the variation of viscosity with temperature is slight and that the Rayleigh number is close to its critical value.

It may be asked why fifth- and higher-order terms in (23) do not contribute many other equilibrium points, and also destabilizing terms outweighing the fourth-order stabilizing terms. This may well be the case for large amplitude disturbances but for $|Y| \ll 1$ and $|Z| \ll 1$ higher-order terms are negligible. The situation is illustrated in figure 2 by comparing possible solutions to the simple differential equation $dy/dt = \sin y$ and the approximate equation $dy/dt = y - \frac{1}{6}y^3$ obtained by truncating the Maclaurin series of $\sin y$. We also mention a discussion by Stuart (1961, p. 139) which indicates that all the terms kept in (23), linear and non-linear, are of the same order of magnitude while those neglected are of higher order.

Two further remarks are necessary.

(i) As in Segel & Stuart (1962), hexagonal solutions to (21) and (22) are stable when $0 < \epsilon < \epsilon^*$ and unstable when $\epsilon > \epsilon^*$, where

$$\epsilon^* \equiv a^2 Q^{-2}(4R + R_1), \quad Q \equiv 2(2R - R_1).$$

The slight lack of precision in the values of R and R_1, due to the possible inclusion therein of higher-order quantities, slightly effects the exact value of ϵ dividing stable from unstable hexagons but does not alter the fact that such an ϵ exists. What happens to hexagonal solutions to (21) and (22) when $\epsilon = \epsilon^*$ can be computed, but the result will generally be altered by higher-order terms—which is immaterial, as the fact that hexagons become unstable, and not exactly how they become so, is what interests us. We have here an example of how neither the neglect of fourth-order terms in the amplitude equations, nor the slight lack of precision in the values of the terms kept, affects the predicted behaviour of solutions to the governing partial differential equations.

(ii) A point which is a stable node in the (Y, Z)-plane does not necessarily represent a stable equilibrium point of the governing partial differential equation: disturbances having a different geometric dependence from those considered may render the solution unstable.

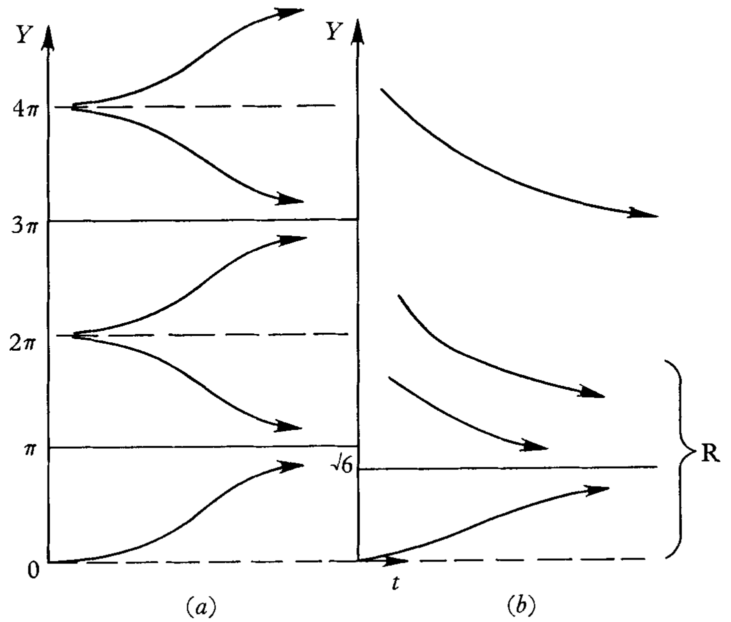

FIGURE 2. An illustration of the correct qualitative behaviour obtained by neglecting higher-order terms in a non-linear differential equation. (Solid horizontal lines indicate stable equilibrium solutions; dotted horizontal lines indicate unstable equilibrium solutions.) (a) Full equation $dy/dt = \sin y$. (b) Truncated equation $dy/dt = y - \frac{1}{6}y^3$. R: region of correct qualitative behaviour.

4. Equation structure sufficient for the appearance of hexagonal cells

The previous sections might interest those with only a passing familiarity with non-linear analyses of cellular convection, but this section is meant for those with a detailed interest in the subject. The model equation was constructed with the aid of what seemed at the time to be an adequate understanding of the important general features of the Boussinesq equations. Further reflexion made it clear that fairly careful investigation would be necessary to uncover the reason why analysis of any sensible model gives rise to amplitude equations whose coefficients have just the proper relationships to insure the presence of hexagonal cells. This investigation is the subject of the present section.

To avoid repetition we concentrate on one representative non-linear portion of the Boussinesq equations; the other non-linear terms, including those arising from the temperature-dependent viscosity, can be dealt with in exactly the same way. Denoting the actual velocity components by $(u, v, w) \equiv \mathbf{V}$, we may write (see I)

$$\mathscr{L}(w) = \Delta Q(u, v, w) + \ldots, \tag{24}$$

where $\mathscr{L}$ is a certain linear operator,

$$Q \equiv U_{xz} + V_{yz} - \Delta_1 W,$$
$$(U, V, W) \equiv (\mathbf{V}.\nabla)(u, v, w), \tag{25}$$

and ... in (24) indicates the presence of other non-linear terms plus linear terms (containing t-derivatives, or proportional to $\mathscr{R} - \mathscr{R}_c$) analogous to those in $\mathscr{M}$ above. Furthermore

$$\Delta_1 u = -w_{xz}, \quad \Delta_1 v = -w_{yz}. \tag{26}$$

We will consider the non-linear modification of the solution associated with a vertical velocity w which according to linear theory has the form

$$w_1 = \phi(x, y, t) f(z), \tag{27}$$

where ϕ satisfies

$$\Delta_1 \phi = -\pi_2 \alpha^2 \phi. \tag{28}$$

The appropriate solutions to (26) turn out to be the simplest

$$u_1 = k\phi_x f', \quad v_1 = k\phi_y f', \quad k \equiv (\pi\alpha)^{-2}. \tag{29}$$

Using these results of linear theory we proceed to the next approximation, which requires solving

$$\mathscr{L}(w_2) = \Delta Q(u_1, v_1, w_1) + \ldots,$$

where

$$Q(u_1, v_1, w_1) = \tfrac{1}{2} d\Delta_1 \{[(kf')^2 - kf^2][\phi_x^2 + \phi_y^2] + [kff'' - f^2]\phi^2\}/dz. \tag{30}$$

We note here that

$$U_y(u_1, v_1, w_1) - V_x(u_1, v_1, w_1) = 0,$$

which is why it is unnecessary in our calculations to retain non-linear terms in (26). We now discuss the following lemma, to be proved later:

As far as the (x, y)-dependence is concerned the coefficients of the second-order replicating terms in ΔQ have the same ratio as the coefficients of the second-order replicating terms in ϕ^2.

To see the significance of this, let

$$\phi = Y(t) \cos \tfrac{\sqrt{3}}{2} \pi\alpha x \cos \tfrac{1}{2} \pi\alpha y + Z(t) \cos \pi\alpha y, \tag{31}$$

the case considered above and in I and II. From the expression for ϕ^2 in (13) and (14), the replicating terms in ϕ^2 then are, according to the lemma,

$$\tfrac{1}{4} C Y^2 \cos \pi\alpha y f(z) \quad \text{and} \quad CYZ \cos \tfrac{\sqrt{3}}{2} \pi\alpha x \cos \tfrac{1}{2} \pi\alpha y f(z), \tag{32}$$

where the constant C depends on the nature of the z dependence. The result of (32) that the YZ coefficient is four times the Y^2 coefficient leads to the fact that the YZ coefficient in the Y' amplitude equation (derived from the Boussinesq equations) is four times the Y^2 coefficient in the Z' amplitude equation. This fact about second-order coefficients in the amplitude equations and similar facts which hold for the third-order coefficients ensure that there are hexagonal equilibrium points [where $Y = \pm 2Z$].

To prove the lemma, we refer to (30) and consider the term in $Q(u_1, v_1, w_1)$ proportional to $\Delta_1 \phi^2$. (We emphasize that for our purposes ΔQ is completely representative of the non-linear terms; no new ideas are required for dealing with the other terms and the results are unchanged.) $\Delta_1 \phi^2$ satisfies what we can call *Property P: The (x,y) dependence of each term is the same as in ϕ^2 and,* the operator Δ_1 multiplying each term having the same overall wave-number by the same constant, *coefficient ratios of terms having the same overall wave-number are preserved but coefficient ratios of terms having different overall wave-numbers are altered.* For example, considering three terms from (14),

$$\Delta_1[\tfrac{1}{4}Y^2\cos\sqrt{3}\,\pi\alpha x+\tfrac{1}{4}Y^2\cos\pi\alpha y+YZ\cos\tfrac{\sqrt{3}}{2}\pi\alpha x\cos\tfrac{1}{2}\pi\alpha y]$$
$$=-3\pi^2\alpha^2[\tfrac{1}{4}Y^2\cos\sqrt{3}\,\pi\alpha x]-\pi^2\alpha^2[\tfrac{1}{4}Y^2\cos\pi\alpha y+YZ\cos\tfrac{\sqrt{3}}{2}\pi\alpha x\cos\tfrac{1}{2}\pi\alpha y].$$

The coefficient ratio of the last two terms, $YZ/\frac{1}{4}Y^2$, is preserved; that of the first two terms is altered. Terms in $\Delta_1(\phi^2)$ which replicate those in ϕ must all have overall wave-number α, as ϕ does, so their coefficient ratios are the same as for the corresponding terms in ϕ^2. This proves the lemma for the term in Q proportional to $\Delta_1\phi^2$. We deal similarly with the other term, proportional to $\phi_x^2+\phi_y^2$, by using the identity

$$\phi_x^2+\phi_y^2=\tfrac{1}{2}\Delta_1(\phi^2)-\phi\Delta_1\phi,$$

from which it is clear that $\phi_x^2+\phi_y^2$ has property P. As operation by Δ preserves property P, the lemma is true. Indeed, it is clear that ΔQ itself has property P.

It is one of the important general features of our solution to the Boussinesq equations that the z-dependence is separated from the (x,y)-dependence. For the amplitude equations to have the second-order terms whose role is so important in I, II, and IV, it is necessary that the first-order z-dependence be replicated at second order. Without variation of viscosity with temperature or a similar asymmetry-producing mechanism this cannot occur. [Referring to (30), if f is even with respect to the centre of the layer, $z=\frac{1}{2}$, then $(f^2)'$, $(ff'')'$, $[(f')^2]'$, being odd cannot replicate f. That the presence of second-order terms in the amplitude equations requires vertical asymmetry was deduced from other considerations by Veronis 1961.]

We proceed to a discussion of the third-order terms in the amplitude equations, using the function ϕ of (31). An $O(b)$ correction to the third-order coefficients is negligible, so from here on we set $b=0$ and therefore can take $f(z)=\sin\pi z$. We must find the complete second-order solution, which has the form

$$w_2=\phi(x,y,t)\sin\pi z+\Phi(x,y,t)\sin 2\pi z,$$
$$u_2=\pi k\phi_x\cos\pi z+\tilde{\Phi}_x\cos 2\pi z,$$
$$v_2=\pi k\phi_y\cos\pi z+\tilde{\Phi}_y\cos 2\pi z,$$

where k is given in (29),
$$\Phi=\sum_{n=0}^{4}C_n\Phi_n \tag{33}$$

and $\tilde{\Phi}$ is the same expression with the constants C_n replaced by different constants $\tilde{C}_n$. The form of Φ is obtained at once from the expression for ϕ^2 in (13) and (14), and the fact that, except for the replicating terms which are dealt with separately, second-order terms in w are found from the undetermined coefficient solutions to a differential equation whose right-hand side, being made up of ΔQ and other terms behaving in the same way, has property P. [$C_0=0$ in (33) because of the Δ_1 factor in (30), but $C_0 \neq 0$ when terms other than ΔQ are considered.]

Proceeding to the third approximation, we now consider the solution to $\mathscr{L}(w_3) = \Delta Q(u_2, v_2, w_2) + \ldots$ and look for third-order replication. A typical third-order term in $\Delta Q(u_2, v_2, w_2)$ is $\Delta(\Delta_1 T)$, where

$$T \equiv [\phi_x \Phi_x + \phi_y \Phi_y] \sin \pi z \equiv T_1 \sin \pi z.$$

Since we are only interested in replicating third-order terms, which have overall wave-number α, the identity

$$\Delta_1(\Phi\phi) = \Phi\Delta_1\phi + \phi\Delta_1\Phi + 2T_1$$

becomes $\quad -\pi^2\alpha^2\phi\Phi = -\pi^2\alpha^2\phi\Phi + \phi\Delta_1\Phi + 2T_1, \quad$ or $\quad T_1 = \frac{1}{2}\phi\Delta_1\Phi.$

The contribution of T to the replicating third-order terms is therefore equal to $\sin \pi z$ times the product of ϕ and a linear combination of the terms in Φ; the constants in Φ must be altered due to the different effect of Δ_1 upon terms of different overall wave-numbers. Once again this behaviour is entirely typical not only of T but of all the non-linear terms: *each replicating third-order term, and therefore the sum of all such terms, is given by a multiple of $\phi\Phi \sin \pi z$ for some constants C_n in Φ.* Computing $\phi\Phi$ from (31) and (33) we find that the replicating terms are, for some constants C_0, C_1, C_3, C_4,

$$Y \cos \tfrac{\sqrt{3}}{2}\pi\alpha x \cos \tfrac{1}{2}\pi\alpha y [Y^2(C_0 + \tfrac{1}{2}C_1 + \tfrac{1}{2}C_3 + \tfrac{1}{4}C_4) + 2Z^2(C_0 + C_1 + C_3)] \sin \pi z$$
$$+ Z \cos \pi\alpha y [Y^2(C_0 + C_1 + C_3) + Z^2(2C_0 + C_4)] \sin \pi z.$$

With the abbreviations

$$R \equiv C_0 + \tfrac{1}{2}C_1 + \tfrac{1}{2}C_3 + \tfrac{1}{4}C_4, \quad R_1 \equiv 2C_0 + C_4, \quad P \equiv 2(C_0 + C_1 + C_3), \tag{34}$$

a = coefficient of YZ in second-order term

replicating $\cos \frac{\sqrt{3}}{2}\pi\alpha x \cos \frac{1}{2}\pi\alpha y f(z)$,

the amplitude equations for Y and Z are therefore identical with equations (21) and (22). The relation $P = 4R - R_1$, necessary and sufficient for the appearance of hexagonal equilibrium solutions, follows at once from (34).

We have shown that the amplitude equations (21) and (22) will be the same for a large class of governing partial differential equations. Two important properties of this class of equations, broadly stated, are:

(i) in the solutions to the linearized equations, the (x, y)-dependence must be capable of being separated out into a function $\phi(x, y)$ satisfying $\Delta_1\phi = -\pi^2\alpha^2\phi$;

(ii) the non-linear terms must roughly be odd-order z-derivatives of squares of linear terms, but, because of (i), repeated appearance of powers of Δ and Δ_1 is permitted.

The appearance and stability of hexagonal cells therefore depends on certain general characteristics of the relevant equations, not on their detailed form. When equations more accurate than those of I, II, and IV, are considered—equations incorporating, for example, more realistic boundary conditions, a less special law of viscosity variation, and a non-constant thermal conductivity—we now have good reason to hope that the qualitative conclusions of I, II, and IV will not be appreciably changed. For confirmation, see Davis (1964).

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